ON CERTAIN SELF-SIMILAR MOTIONS OF GAS IN THE PRESENCE OF SHOCK AND DETONATION IN A MEDIUM OF VARYING DENSITY

(O NEKOTORYKH AVTOMODEL'NYKH DVIZHENIIAKH GAZA PRI UDARE I DETONATSII V SREDE S PEREMENNOI PLOTNOST'IU)

PMM Vol.30, № 6, 1966, pp. 1133-1139

E.I. ANDRIANKIN

(Moscow)

(Received March 8, 1966)

Self similar motions of gas subject to shock by a piston were analyzed by Sedov [1]. Equations derived there were applied not only to problems concerning pistons [2 to 4], but also to the mathematical analysis of detonation propagation in gases of variable density.

An analysis is made below of two-dimensional shock and detonation waves in gases the density of which varies in accordance with the law $\rho_0 = a_0 x_0^{\omega}$ (x_0 is the initial coordinate of a particle), and its unit mass heat content to $Q = Q_0 x_0^{2/3}$, in contrast to [5 and 6] where Q = const .

We also assume that behind the detonation wave there is a piston moving with a velocity $\sim t^{\beta/1-\beta}$.

The presence of the additional parameter β considerably complicates the field of integral curves as compared, for example, to the case considered in [5], due to the simultaneous inclusion in the analysis of a strong explosion, a piston, a solution of a dipolar type, a detonation from the free boundary and of a short shock [7].

The behavior of integral curves is analyzed.

Cases are noted in which the detonation from the free boundary must nevertheless be supercompressed, while in the presence of shock it must correspond to the Jouguet point. An exact solution is found for the problem of a short shock in a medium of varying density for a number of values of ω dependent on γ , thus generalizing the solution of the known case of $\gamma = 1.4$, ρ_0 = const .

1. The motion of a gas behind a strong detonation wave (or shock wave, when $Q_0 = 0$) are defined by the following equations of hydrodynamics

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_0}, \qquad \frac{\partial u}{\partial x_0} = \frac{\partial}{\partial t}\frac{\rho_0}{\rho}, \qquad p\rho^{-\gamma} = p_*(x_0)\rho_*^{-\gamma}(x_0) \tag{1.1}$$

In the following text the subscript $*$ will always be used for denoting parameter values at the wave front. Boundary conditions for (1.1) are determined by the conservation laws at the front $x_*(t)$

$$\rho_0 D = \rho_*(D - u_*), \qquad p_* = \rho_0 D u_*$$

$$\frac{(D-u_*)^2}{2} + \frac{\gamma p_*}{\rho_*(\gamma-1)} = \frac{D^2}{2} + Q_0 x_*^{2/3}, \qquad D = \frac{dx_*}{dt} \tag{1.2}$$

and by the equality of gas and piston velocities $u(0,\ t) = v_0 x_*$, or in the case of expansion into vacuum, by the pressure being equal to zero, $p(0,t)=0$.

We seek a solution in the form of

$$u = u_* v(x), \qquad p = p_* f(x), \qquad \rho = \rho_* q(x), \qquad x = x_0 / x_*$$
$$u_* = u_0 x_*^{\beta}, \quad p_* = a_0 D_0 u_0 x_*^{2\beta+\omega}, \qquad D = D_0 x_*^{\beta}, \qquad \rho_* = \frac{a_0 D_0}{D_0 - u_0} x_*^{\omega} \qquad (1.3)$$

Boundary conditions (1.2) are satisfied when constants D_0, u_0 and Q_0 are related by Expressions

$$u_0^2 \left(\frac{2\gamma}{\gamma - 1} - \lambda\right) = 2\lambda Q_0, \qquad \lambda = \frac{\gamma u_0}{D_0 - u_0} \qquad (1.4)$$

In the physical sense λ represents the squared ratio of the velocity of sound to the mass flow rate behind the wave front in a coordinate system in which the front is at rest.

The value of parameter Q_0 is assumed to be specified, while λ is defined as the eigen number of the boundary value problem with $x_0 = 0$ and $x_0 = x_*$, i.e. is dependent on the piston velocity.

Substituting (1.3) into (1.1) we find the system of equations of dimensionless functions

$$x^{-\omega} \frac{df}{dx} - x \frac{dv}{dx} + \beta v = 0, \qquad \lambda \frac{dv}{dx} + \gamma x \frac{d}{dx} x^{\omega} q^{-1} = 0$$
$$f = q^{\gamma} x^{\mu}, \quad \mu = 2\beta + \omega - \gamma\omega, \quad f(1) = 1, \quad v(1) = 1 \qquad (1.5)$$

Equations (1.5) allow a similarity set. After the change of variables

$$f^{\gamma+1} = \left(\frac{z}{\lambda}\right)^{\gamma} x^{\delta}, \quad \left(\frac{v}{w}\right)^{\gamma+1} = \left(\frac{z}{\lambda}\right)^{\gamma} x^{2\beta+\gamma-1} \qquad (\delta = \omega(\gamma+1) + 2\gamma + 2\beta) \quad (1.6)$$

they become

$$(1 - z)\frac{dz}{d\eta} = \varphi(z, w), \quad (1 - z)\frac{dw}{d\eta} = \psi(z, w), \quad \eta = \ln x, \quad z(1) = 1, \quad w(1) = 1 \quad (1.7)$$

Eliminating the variable η, we obtain

$$\frac{dz}{dw} = \frac{\varphi}{\psi}, \qquad \varphi = 2z\left\{\beta - 1 + [\delta + (\gamma+1)\beta w]\frac{z}{2\gamma}\right\}$$
$$\psi = \omega + 2\beta + (1 - \beta) w - (\omega + 1) wz - \beta w^2 z, \qquad z|_{w=1} = \lambda \qquad (1.8)$$

These expressions contain four parameters γ, β, ω and λ which determine the solution character.

We shall limit the area of variation of these parameters for physical reasons.

We shall consider detonation and shock waves which satisfy the necessary stability conditions $c_* + u_* \geqslant D$, $c_*^2 \rho_* = \gamma p_*$, when small perturbations are catching up with a strong discontinuity front. It follows from (1.4) that this condition is fulfilled if $\lambda \geq 1$. On the other hand, we have $\lambda \leq 2\gamma/(\gamma - 1)$, because $Q_0 \geq 0$. It is known from thermodynamics that $c_p - c_v > 0$, consequently $\gamma > 1$.

The condition for the mass of matter behind the wave front to be finite stipulates $\omega > -1$. We note that when $\beta \geq 1$, a shock wave cannot detach itself from the coordinate origin, even after an infinitely long period of time. Consequently, there exists a self-similar solution when $\beta < 1$. In the wave front proximity there is an area $x_1 < x \leq 1$, where $u > 0$, because $D > 0$. With this, the total mass of gas either moves in the direction of $x > 0$, or $u(x_1) = 0$. The impulse of the mass of gas moving in the direction of $x > 0$ is proportional to $x^{\omega+1+\beta}$ and, because $p(x,\ t) \geq 0$, it does not diminish with time (see [8],*p.605), therefore, $\omega + \beta + 1 \geq 0$. It will be shown in Section 3 that, if in the latter expression the equality sign is valid, there will be no continuous solutions in the whole area defined by $0 \leq x \leq 1$. We shall therefore assume that $\omega + \beta + 1 > 0$. Continuous solutions are also limited by variations of parameter $\beta \geq \beta_0 <$ $< -0.5(\omega + 1)$ up to the value of β_0 corresponding to a short shock.

The area of variation of z and w is also limited. As pressure is a nonnegative parameter, we have $z \geq 0$. Simplifying Equation (1.7) in the neighborhood of the wave front, where $w \to 1$ and $0 < dw/d\eta = w + \beta + 1 + O(w-1)$, we come to the conclusion that the value of $w \leq 1$ must decrease with decreasing x.

We write down the derived limitations

$$z \geqslant 0,\ w \leqslant 1,\ \gamma > 1,\ \omega + 1 > 0$$
$$\omega + 1 + \beta > 0,\ \beta < 1,\ 1 \leqslant \lambda \leqslant 2\gamma / (\gamma - 1) \qquad (1.9)$$

2. We shall plot the field of integral curves of Equation (1.8). The isoclines of zeros $z_0(w)$, infinities $z_\infty(w)$, and the particular solution $z \equiv 0$, all intersect at the following singular points

$$A\left(w_1 = \frac{\omega + 2\beta}{\beta - 1},\quad z_1 = 0\right)$$

$$B\left(w_2 = -2 - \frac{\omega}{\beta},\quad z_2 = 1\right), \qquad C\left(w_3 = \frac{\omega(\gamma - 1) + 2\gamma + 2\beta}{(\gamma - 1)(1 - \beta)},\quad z_3 = \frac{2\gamma(1 - \beta)}{w_3(2\beta + \gamma - 1)}\right)$$

Point A is always a "saddle" point. For the isocline of infinities $z_\infty = z_\infty(w)$, as well as for the slope of the particular solution $z_1(w)$ at point A we have

$$\frac{dz_\infty}{dw} = \frac{(1 - \beta)^3}{(2\beta - 1)(\omega + 2\beta)(\omega + 1 + \beta)}, \qquad \frac{dz}{dw} = \frac{1 - \beta + 2(1 - \beta)^3}{(2\beta - 1)(\omega + 2\beta)(w + 1 + \beta)}$$

In the neighborhood of B, Equation (1.8) becomes

$$\frac{dy}{dr} = \frac{ar + by}{cr + ey}, \quad y = z - z_2, \quad r = w - w_2, \quad \gamma a = (\gamma + 1)\beta z_2^2 \qquad (2.1)$$

$$b = 2(1 - \beta), \qquad c = 1 - \beta - (\omega + 1) z_2 - 2\beta w_2 z_2, \qquad e = -w_2(\omega + 1 + \beta w_2)$$

The singular point B will be a "saddle" point when $ae - bc > 0$; a nodal point, when inequality $0 < l < (b + c)^2$ is fulfilled with $l = 4(ae - bc) + (b + c)^2$, and a focal point when $l < 0$.

The slope of the zero and infinity isoclines at point B will be

$$\left.\frac{dz_0}{dw}\right|_2 = \frac{(\gamma + 1)\beta}{2\gamma(\beta - 1)}, \qquad \left.\frac{dz_\infty}{dw}\right|_2 = \frac{(\omega + 3\beta)\beta}{(2\beta - 1)(\omega + 2\beta)} \qquad (2.2)$$

and the slope of singular solutions are found from (2.1), if $y = Kr$ is assumed.

In the neighborhood of C Equation (1.8) takes the form of (2.1), if in the latter we substitute w_3 and z_3 for w_2 and z_2, respectively. For analyzing this problem at points B and C it is conveneient to express the term $(ae - bc)$ in the form

$$\gamma(ae - bc)_2 = (\gamma - 1)\beta(1 - \beta)(w_2 - w_3) \qquad (2.3)$$

$$(2\beta + \gamma - 1)(ae - bc)_3 = 2(\gamma - 1)\beta(1 - \beta)^2 w_3 (w_3 - w_2)$$

The analysis, with inequalities (1.9) taken into account, is reduced to the consideration of ten characteristic cases shown in Fig.1, where curves of identical physical meaning are denoted by the same letters. Only the isoclines essential for the analysis of this problem have been shown (by dotted lines) in Fig.1. Possible directions of solutions are indicated by arrows. Points on segment KL correspond to conditions at the wave front, with points K (the Jouguet point), $L(Q = 0)$, and K_1 (supercompressed detonation) defined by coordinates

$$K(w = 1:\ z = 1), \quad L\left(w = 1,\ z = \frac{2\gamma}{\gamma - 1}\right), \qquad K_1(w = 1,\ z = \lambda > 1)$$

The asymptotic behavior of curves between LQ and KN (or K_1N) for $z \to \infty$, $w \to 0$ is found from (1.8)

$$z \sim |w|^{-\alpha} \sim x^{-\alpha(w+1)}, \quad \gamma(\omega + 1)\alpha = \omega(\gamma + 1) + 2\gamma + 2\beta \qquad (2.4)$$

$$f(0) \to \text{const}, \qquad v(0) \to \text{const}$$

These curves, therefore, correspond to a flow pattern in which the piston moves in the direction of $x > 0$, or $x < 0$, depending on whether it approaches the z-axis from the left, or right, respectively.

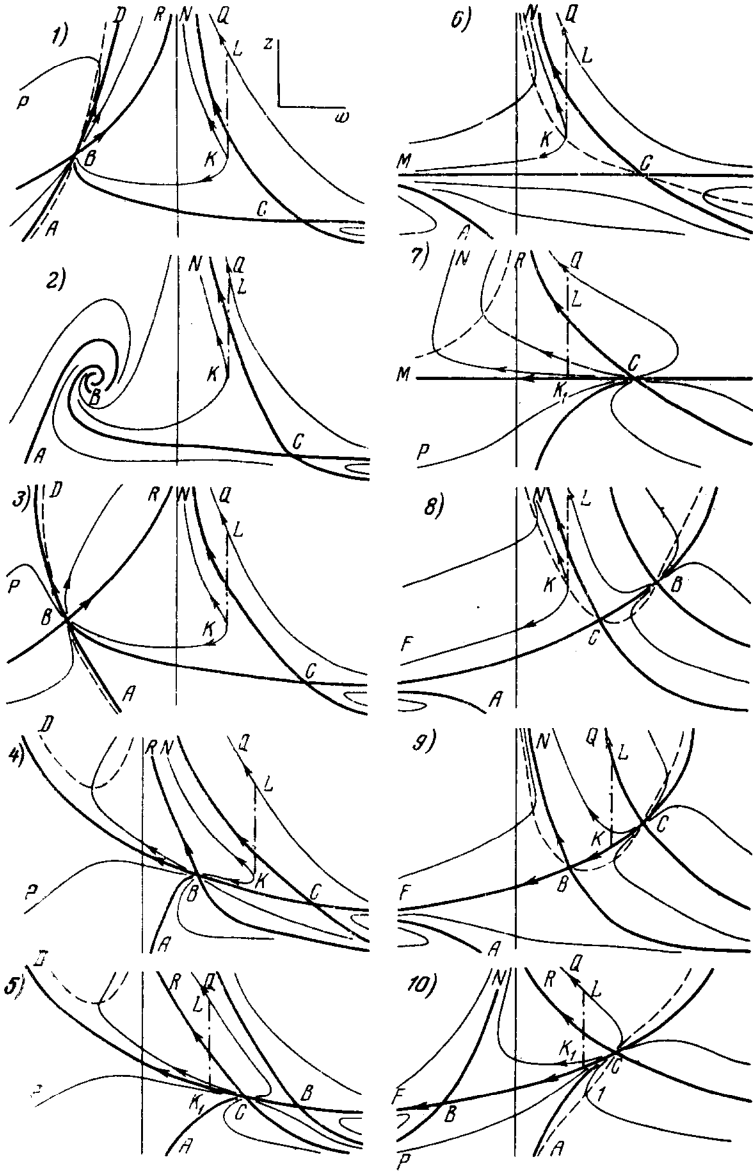

Fig. 1

Curve LQ corresponds to the compression of the nondetonating gas. The asymptotic behavior of curves between DR is also defined by (2.4), with $v(0) < 0$, The asymptotic behavior of curves BR and CR is determined by

$$z = \frac{K_1}{w} \sim x^{-\alpha(\omega+1)}, \qquad (\omega + \gamma + 2\beta) K_1 = -\omega - 2\beta$$
$$f \to \text{const}, \qquad v \sim x^{(\omega+1)(\alpha-1)} \to 0 \tag{2.5}$$

which means that these correspond to a detonation from the closed end, or to a strong explosion.

When $z \to \infty$, $w \to w_4 = -(\omega + 1)/\beta$, the behavior of curves BD, or CD is defined by

$$z = \frac{K_2}{w - w_4} \sim x^{-\varepsilon}, \qquad \gamma\varepsilon = 2\beta + \gamma - 1, \qquad \beta(1+\varepsilon) K_2 = (\omega + 1)(1 - 2\beta)$$
$$f \sim x^{\omega+1} \to 0, \ v \to \text{const} < 0 \tag{2.6}$$

and correspond to a detonation from the free boundary.

The asymptotic behavior of KF or K_1F for $w \to -\infty$ has the form (2.7)

$$z = \frac{\gamma(1-\beta)}{\beta w} \sim x^{(1-\beta)(\gamma-1)}, \ f \sim x^q \to 0, \ v \sim -x^\beta, q = \omega + \beta + 1 + (\gamma - 1)(1 - \beta), \ \beta < 0$$

and corresponds to a discharge into vacuum.

Integral curves corresponding to continuous solutions cannot intersect line $z = 1$ at any point $w = w_5$, except where $w_5 = w_2$. We find from (1.7) that in the neighborhood of $z = 1$, $w = w_5$, we have $d(w - w_s)^3/d\eta > 0$. This means that, once line $z = 1$ had been crossed, x begins to grow again, and $x \to \infty$ with $w \to -\infty$, which contradicts physical notions.

Fig.1 shows the field of integral curves for all possible cases of variation of parameters γ, ω, β and λ which satisfy inequalities (1.9)

Case 1. $0.5 \le \beta < 1$, $l_2 = (\omega + 5\beta - 2)^2 - 4(1 + 1/\gamma)(2\beta - 1)(\omega + 2\beta) \ge 0$. In accordance with (2.3), $(ae - bc)_2 < 0$, and as also $l_2 \ge 0$, therefore, B is the nodal point. The solution emanating from point B is below the isocline of zeros. The form of Equation (1.7) in the neighborhood of point B is

$$k\frac{dw}{d\eta} = A_1 + kA_2, \qquad \left(\frac{dz_\infty}{dw}\right)_2 = -\frac{A_1}{A_2}, \qquad k = \left(\frac{dz}{dw}\right)_2 \tag{2.8}$$

$$A_1 = \omega + \beta + 2\beta w_2, \qquad A_2 = (\omega + 1 + \beta w_2) w_2$$

Because $k < dz_\infty/dw$, it follows from (2.8) that $dw/d\eta < 0$, which means that the solution can be extended in these directions. It will be seen from Fig.1 that the Jouguet point corresponds not only to detonation from the open end, but also to the case of the piston moving with velocity greater than a certain given value $v(0) \le v_1 > 0$. The asymptotic behavior of curve KN is in accordance with (2.4).

Case 2. $0.5 < \beta < 1$, $l_2 < 0$. B is a focal point. When approaching this point, the curves intersect line $z = 1$ at points $w \ne w_2$, while x becomes nonsingle-valued. Continuous solutions, can, therefore, exist only when the velocity of the compressing piston is $v(0) > v_0 > 0$.

Case 3. $0 < \beta < 0.5$, $\omega + 2\beta \ge 0$. B is a nodal point, because of $w_2 - w_3 < 0$, while the minimum value of $l_2 = 8\beta(1 - 2\beta) > 0$ (for $\gamma \to 1$ and $\omega = 3\beta - 2$).

Case 4. $0 < \beta < 0.5$, $-\delta \le \omega + 2\beta < 0$, $(2\beta + \gamma - 1)\delta = 2\gamma\beta(1 - \beta)$. B (a nodal point) moves into the right-hand half-plane. The Jouguet point corresponds to detonation from the free boundary (2.6), as well as to detonation with piston velocity $v(0) \le v_1 \ge 0$.

Case 5. $0 < \beta < 0.5$, $\omega + 2\beta < -\delta$. Points B and C exchange their respective places. In accordance with (2.3), C is not a "saddle" point, and neither is it a focal point in view of

$$l_3 / x^2 = (a_1 + b_1 + c_1)^2 - 4\,(1 + 1/\gamma)\, a_1 b_1 > (a_1 + 2b_1)^2 - 8a_1 b_1 > 0$$
$$c_1 = 1 - \beta > a_1 = 3 z_3 w_3 > 0, \qquad c_1 > b_1 = z_3\,(\omega + 1 + \beta w_3) > 0$$

Therefore, C is a nodal point. Possible continuous solutions are bounded by curve $K_1 D$ which corresponds to supercompressed detonation from the open end (2.6).

Case 6. $\beta = 0$, $\omega > 0$. Point B has receded into infinity, and C is a "saddle" point. The particular solution $z \equiv \lambda_0$, $[(\gamma + 1)\omega + 2\gamma]\lambda_0 = 2\gamma$ is unstable, since $\lambda_0 < 1$. The asymptotic behavior of curves KM is defined by

$$z \to \lambda_0, \qquad w \to -\infty, \qquad f \sim x^{\omega + 2\gamma/(\gamma+1)} \to 0, \qquad v \to \text{const} < 0 \tag{2.9}$$

This case represents detonation from the open end.

Case 7. $\beta = 0$, $-1 < \omega \leq 0$. C is a nodal point. The particular solution is $z \equiv \lambda_0 \geq 1$, and the detonation, even from the open end ($K_1 M$), is supercompressed.

Case 8. $\omega + 2\beta \geq -\beta > 0$. The Jouguet point corresponds to detonation from the free boundary, or to compression by piston with $u(0, t) > 0$. The two singular points B and C are located to the right of line KL , and it follows from (2.3) that:

a) if $\omega + 2\beta \geq -\delta$ and $2\beta + \gamma - 1 > 0$, then $w_2 \geq w_3 > 1$, B is a nodal point, and C a "saddle" point. If the equality sign applies, these two points coincide.

b) if $-\beta < \omega + 2\beta < -\delta$, $2\beta + \gamma - 1 > 0$, then $w_3 > w_2 \geq 1$, and C is a nodal point, while B a "saddle" point.

c) if $\omega + 2\beta \geq -\beta$, $2\beta + \gamma - 1 \leq 0$, then $w_2 \geq 1$, B is a "saddle" point, and C moves into the lower half-plane.

Case 9. $0 \leq \omega + 2\beta < -\beta > 0$. B is a "saddle" point ($0 < w_2 < 1$) . The separation line from point B runs above $z = 1$ for $w = 1$. Continuous solutions are, therefore, above the Jouguet point. Detonation is supercompressed even from the free boundary ($K_1 BF$). In accordance with (2.8), solutions from B may only be extended in the directions indicated by arrows. C ($w_3 > 1$) is a nodal point when $2\beta + \gamma - 1 > 0$. If $2\beta + \gamma - 1 \leq 0$, then C moves into the lower half-plane.

Case 10. $\omega + 2\beta < 0$, $\beta < 0$. B is a "saddle" point, C a nodal point, $w_2 < 0$, $w_3 > 1$. With $2\beta + \gamma - 1 > 0$, $z_3 > 0$, when $2\beta + \gamma - 1 \leq 0$, point C moves into the lower half-plane.

Depending on the law of variation of the total gas energy, which is proportional to x_*^{ν}, with $\nu = \omega + 2\beta + 1$, we obtain the following characteristic cases:

a) With $\nu > 0$, a detonation (supercompressed) from point K_1 is possible with a free boundary $K_1 BF$ and a piston moving in the negative direction of axis $x K_1 BN$.

b) When $\nu = 0$, curve LQ , which now coincides with the separation line CR , corresponds to a strong explosion. Curves below line LQ correspond to detonation in the presence of a piston which for $t \to 0$ absorbs a logarithmically infinite part of the detonating mixture energy, while the total energy of the gas remains constant. A discharge into vacuum $K_1 BF$ is also possible with an infinite energy (2.8), but in contrast to [7] this takes place when $Q_0 > 0$.

c) With $\nu < 0$, curve LQ runs below the separation line CR . The piston imparts to the gas an infinitely great energy by shock $t = 0$), and immediately begins to take it away by reversing its motion. At any finite instant of time the energy is also finite, even when $Q_0 > 0$ (curves between LQ and $K_1 BN$). These solutions bring to mind those of the dipole type for nonlinear thermal conductivity [9].

d) When $\nu = \nu_0 < 0$, point L descends lower and lower with decreasing ν, finally reaching point K_1 of line CF. In this case there exists a solution of either the dipole type (in the presence of a piston $u(0, t) < 0$), or of the impulse type K_1PF when the gas energy is infinitely great.

e) With $\nu < \nu_0$, there are no continuous solutions in the whole area defined by $0 \le x \le 1$, since in this case curves CP intersect line $z = 1$ at points $w \ne w_2$. We may, however, visualize a short shock with energy absorption at the front $\lambda > 2\gamma/(\gamma - 1)$. With $\beta > 1$ the self-similar solutions may be considered as a limit for the non-self-similar solutions (if, for example, $Q = Q_* + Q_0 x_0^{2\beta}$). It appears that in this case continuous solutions are only possible, if the piston velocity $u(0, t) > v > 0$ is greater than a certain specified value.

3. Particular solutions of Equations (1.5) may be derived, for example, in cases of constant wave velocity, existence of energy and impulse integrals, and when the pressure is a power function of x.

Equation (1.8) becomes linear for $\beta = 0$, and is expressed in terms of z by a quadratic form, while the dependence of z on x, derived from (1.7), is

$$w \sqrt{\frac{z}{\lambda}} \left|\frac{z-\lambda_0}{\lambda-\lambda_0}\right|^{\varkappa} = \frac{\omega\lambda_0}{2\lambda} \int_\lambda^z \sqrt{\frac{\lambda}{z}} \left|\frac{z-\lambda_0}{\lambda-\lambda_0}\right|^{\varkappa} \frac{dz}{z-\lambda_*} + 1$$

$$zx = \lambda \left|\frac{z-\lambda_0}{\lambda-\lambda_0}\right|^{1-\lambda_0}, \qquad \lambda_0 = \frac{2\gamma}{2\gamma + \omega(\gamma+1)}, \qquad \varkappa = \frac{1}{2}(w+1)\lambda_0 - \frac{1}{2} \tag{3.1}$$

The integral curves $z = z(w)$ of Equation (3.1) are illustrated in Fig.1 by cases 6 and 7. With $\omega \le 0$, there exists a further particular solution which identically satisfies (1.8), while the definite integral of (1.7) is of the form

$$z \equiv \lambda_*, \qquad x = \left|\frac{\omega + w(1-\alpha\lambda_0)}{\alpha(1-\lambda_0)}\right|^{\frac{1-\lambda_0}{1-\alpha\lambda_0}}, \qquad \alpha = \omega + 1$$

$$f = x^{\frac{2\gamma\lambda_*}{\gamma+1}}, \qquad v = \frac{2\gamma}{(\gamma-1)\lambda_0} x^{\frac{\gamma-1}{\gamma+1}} - \frac{\gamma+1}{\gamma-1}\alpha \tag{3.2}$$

This solution corresponds to supercompressed detonation from the open end (2.9). If $\omega = 0$, then Equation (1.5) has a further solution $v =$ const, $f =$ const which may be matched to (3.2). This case was the subject of detailed studies in [1, 10 and 11].

With $\omega + 2\beta + 1 = 0$, and the piston energy taken into account, the expression of the energy integral [5] which, incidentally, also exists in the case of dipole type of flows, is as follows:

$$(fx)^{1-1/\gamma} + \frac{\gamma-1}{2\gamma}\lambda v^2 x^{\omega+1} - \frac{\gamma-1}{\gamma}\lambda v f + \frac{\gamma-1}{\gamma}\lambda v_0 f_0 = 0 \tag{3.3}$$

$$u(0, t) = u_0 v_0 x_*, \qquad p(0, t) = a_0 D_0 u_0 f_0 x_*^{\omega+2\beta}$$

By virtue of conditions at the wave front we have $2v_0 f_0 = 1 - 2\gamma/(\gamma-1)\lambda$. With known piston velocity $u(0, t)$ and Q_0 we integrate the second of Equations (1.5) by using (3.3), and obtain a single-valued solution from which the value of the constant energy of gas is derived.

A particular solution may also be obtained by assuming that the pressure is a power function of the coordinate. It follows from (1.5) that function f satisfies Equation

$$\frac{d}{dx}\left[x^{-\omega}\frac{df}{dx} + \frac{\gamma}{\lambda}x^2\frac{d}{dx}(x^{\omega+2\beta}f^{-1})^{1/\gamma}\right] = \frac{\gamma\beta}{\lambda}x\frac{d}{dx}(x^{2\beta+\omega}f^{-1})^{1/\gamma}$$

$$f(1) = 1, \qquad (1-\lambda)\left.\frac{df}{dx}\right|_1 = 2\beta + \lambda\beta + \omega \tag{3.4}$$

If $f = x^{\mu}$, then, in accordance with assumptions made in [12], it follows from (3.4) that

$$(\omega + 2\beta - \mu)(\gamma\beta + \mu - \gamma - \omega - 2\beta)\, x^{(\omega+2\beta-\mu)/\gamma} = \mu(\mu - \omega - 1)\, x^{\mu-\omega-2} \qquad (3.5)$$

$$\lambda\mu + \lambda\beta + \omega + 2\beta = \mu \qquad (3.6)$$

Equation (3.5) may be satisfied either by equating to zero coefficients at the various powers of x, or by stipulating the equality of exponents and equating to zero the sum of coefficients. Condition $\omega > -1$, together with Equation (3.6) can only be satisfied by the first procedure, therefore

$$f = x^{\mu}, \qquad v = \left(1 + \frac{\mu}{\beta}\right) x^{\beta} - \frac{\mu}{\beta}, \qquad \mu = \omega + 1, \qquad \beta = \frac{1-\gamma}{2-\gamma}$$

$$\omega = \frac{(3-2\gamma)\lambda - \gamma}{\lambda(\gamma - 2)}, \qquad 1 < \gamma < 2 \qquad (3.7)$$

With $\lambda = 2\gamma/(\gamma - 1)$ we have an exact solution of the problem of impulse in a variable density medium, which is a generalized solution of the known case [13], when $\omega = 0$.

When $1 \le \lambda < 2\gamma/(\gamma - 1)$, then (3.7) defines the motion of a gas in the presence of detonation.

It can be ascertained with the use of (1.6) that with $x_2^{\beta} = \lambda$ this solution passes through point B, Fig.1 (Case 10).

When $\omega + \beta + 1 = 0$; then the second of Equations (1.7) yields the integral (of impulse [5]), $w \equiv 1$, while the first of Equations (1.7) is integrated in the quadratic form

$$x^{2(1-\beta)} z = \lambda \left| \frac{z + b_0}{\lambda + b_0} \right|^{1+b_0}, \qquad b_0 = \frac{2\gamma(\beta - 1)}{2\beta + \gamma - 1}$$

This solution corresponds to curves CP in Fig.1 and cannot be continuous for all $0 \le x \le 1$. In the discontinuity area, such a solution may be arrived at by, for example, artificially bleeding the gas into a container [12] which moves in accordance with a stipulated law.

BIBLIOGRAPHY

1. Sedov, L.I., O nekotorykh neustanovivshikhsia dvizheniiakh szhimaemoi zhidkosti (On certain unsteady motions of a compressible fluid). *PMM* Vol.9, № 4, 1945.
2. Krasheninnikova, N.L., O neustanovivshemsia dvizhenii gaza, vytesniaemogo porshnem (On the unsteady motion of gas expelled by a piston). Izv.Akad.Nauk SSSR, OTN, № 8, 1955.
3. Kochina, N.N. and Mel'nikova, N.S., O neustanovivshemsia dvizhenii gaza, vytesniaemogo porshnem, bez ucheta protivodavleniia (On the unsteady motion of gas driven outward by a piston, neglecting the counterpressure). *PMM* Vol.22, № 4, 1958.
4. Grigorian, S.S., Zadacha Koshi i zadacha o porshne dlia odnomernykh neustanovivshikhsia dvizhenii gaza (Cauchy's problem and the problem of a piston for one-dimensional, unsteady motions of gas). *PMM* Vol. 22, № 2, 1958.
5. Sedov, L.I., Metody podobiia i razmernosti v mekhanike (Methods of Similarity and Dimensional Analysis in Mechanics). 5th edn., M., "Nauka", 1965.
6. Iavorskaia, I.M., Reshenie nekotorykh zadach o detonatsii v srede s peremennoi plotnost'iu (The solution of certain problems of detonation in media of variable density). Dokl.Akad.Nauk SSSR, Vol.111, № 4, 1956.
7. Zel'dovich, Ia.B., Dvizhenie gaza pod deistviem kratkovremennogo davleniia (udara) (Motion of a gas subject to a momentary pressure (shock)). Acoust.J., Vol.2, № 1, 1956.

8. Zel'dovich, Ia.B. and Raizer, Iu.P., Fizika udarnykh voln i vysokotemperaturnykh gidrodinamicheskikh iavlenii (Physics of Shock Waves and High Temperature Gas Dynamics Phenomena). M., Fizmatgiz, 1963.

9. Zel'dovich, Ia.B. and Barenblatt, G.I., O reshenii tipa dipolia v zadachakh nestatsionarnoi fil'tratsii gaza pri politropicheskom rezhime (On the dipole type solutions of problems of nonstationary gas filtration under polytropic conditions). *PMM* Vol.21, № 5, 1957.

10. Zel'dovich, Ia.B. and Kompaneets, A.S., Teoriia detonatsii (Theory of Detonation). M., Gostekhizdat, 1955.

11. Staniukovich, K.P., Neustanovivshiesia dvizheniia sploshnoi sredy (Unsteady Motions of Continuous Media). M., Gostekhizdat, 1955.

12. Adamskii, V.B. and Popov, N.A., Dvizhenie gaza pod deistviem davleniia na porshne, izmeniaiushchegosia po stepennomu zakonu (Motion of a gas under the action of a piston whose pressure varies in accordance with a power law). *PMM* Vol.23, № 3, 1959.

13. Zhukov, A.I. and Kazhdan, Ia.M., Dvizhenie gaza pod deistviem kratkovremennogo impul'sa (Motion of a gas subject to a momentary impulse). Acoust.J., Vol.2, № 4, 1956.

Translated by J.J.D.